

# Symplectic Dirac Equation

R. G. G. Amorim<sup>a,b</sup>, S. C. Ulhoa<sup>a</sup>, Edilberto O. Silva<sup>c</sup>

<sup>a</sup> Instituto de Física, Universidade de Brasília, 70910-900, Brasília, Distrito Federal, Brazil

<sup>b</sup> Faculdade Gama, Universidade de Brasília, Setor Leste (Gama), 72444-240, Brasília, Distrito Federal, Brazil

<sup>c</sup> Departamento de Física, Universidade Federal do Maranhão, Campus Universitário do Bacanga, 65085-580, São Luís, Maranhão, Brazil

## Abstract

Symplectic unitary representations for the Poincaré group are studied. The formalism is based on the noncommutative structure of the star-product, and using group theory approach as a guide, a consistent physical theory in phase space is constructed. The state of a quantum mechanics system is described by a quasi-probability amplitude that is in association with the Wigner function. As a result, the Klein-Gordon and Dirac equations are derived in phase space. As an application, we study the Dirac equation with electromagnetic interaction in phase space.

**Keywords:** Poincaré group, Wigner function, noncommutativity

## 1. Introduction

The first formalism to quantum mechanics in phase space was proposed by Wigner in 1932 [1]. He was motivated by the problem of finding a way to improve the quantum statistical mechanics, based on the density matrix, to treat the transport equations for superfluids [2–4]. Since then, the formalism proposed by Wigner has been applied in different contexts, such as quantum optics [5, 6], condensed matter [7–9], quantum computing [10–12], quantum tomography [13], plasma physics [14–19]. Wigner introduced his formalism by using a kind of Fourier transform of the density matrix,  $\rho(q, q')$ , giving rise to what in nowadays called the Wigner function,  $f_W(q, p)$ , where  $(q, p)$  are coordinates of a phase space manifold  $(\Gamma)$ . The Wigner function is identified as a quasi-probability density in the sense that  $f_W(q, p)$  is real but not positive defined, and as such cannot be interpreted as a probability. However, the integrals  $\sigma(q) = \int f_W(q, p) dp$  and  $\sigma(p) = \int f_W(q, p) dq$  are distribution functions [1, 2].

In Wigner formalism each quantum operator  $A$  in the Hilbert space is associated with a function  $a_W(q, p)$  defined in  $\Gamma$ . The application  $\Omega_W : A \rightarrow a_W(q, p)$  is such that associative algebra of operators in  $\mathcal{H}$  defines an associative but noncommutative algebra in  $\Gamma$ . The noncommutativity stems from nature of the product between two operators in  $\mathcal{H}$ . Given two operators  $A$  and  $B$ , we have the mapping  $\Omega : AB \rightarrow a_W(q, p) \star b_W(q, p)$ , where the star (or Moyal)-product  $\star$  is defined by [20]

$$a_W(q, p) \star b_W(q, p) = a_W(q, p) \exp\left[\frac{i}{2}\left(\overleftarrow{\frac{\partial}{\partial q}} \overrightarrow{\frac{\partial}{\partial p}} - \overleftarrow{\frac{\partial}{\partial p}} \overrightarrow{\frac{\partial}{\partial q}}\right)\right] b_W(q, p). \quad (1)$$

(Throughout this Letter we use natural units:  $\hbar = c = 1$ ). Note that Eq.(1) can be seen as an operator  $\hat{A} = a_W(q, p) \star$  acting on functions  $b_W(q, p)$ , such that  $\hat{A}(b_W) = a_W \star b_W$ . In this sense, we can study unitary representations of Lie groups in phase space using the Moyal product as defined by the operators  $\hat{A}$ . This gives rise, for instance, to the Klein-Gordon and Dirac equations written in phase space [21–24]. The connection with Wigner function is derived, providing a physical interpretation for the representation. As a consequence, these symplectic representations are a way to consider the Wigner methods on the bases of symmetry groups. In the present work, we apply this symplectic formalism to solve Dirac equation with electromagnetic interaction in phase space. These results provide a starting point for our analysis of nonclassical electromagnetic radiation states in phase space.

The presentation of this Letter is organized in the following way. In section 2, we define a Hilbert space  $\mathcal{H}(\Gamma)$  over a phase space  $\Gamma$  with its natural relativistic symplectic structure. In section 3, we study the Poincaré algebra in  $\mathcal{H}(\Gamma)$  and the representation for spin 1/2. In section 4, the Dirac equation in phase space with electromagnetic radiation is considered. Quasi-amplitudes of probabilities are derived. In section 5, final concluding remarks are presented.

## 2. Hilbert Space and Symplectic Structure

Consider  $M$  an  $n$ -dimensional analytical manifold where each point is specified by Minkowski coordinates  $q^\mu$ , with  $\mu = 0, 1, 2, 3, 4$  and metric specified by  $diag(g) = (- + + +)$ . The coordinates of each point in  $T^*M$  will be denoted by  $(q^\mu, p^\mu)$ . The space  $T^*M$  is equipped with a symplectic structure by introducing a 2-form

$$\omega = dq^\mu \wedge dp_\mu, \quad (2)$$

Email addresses: ronniamorim@gmail.com (R. G. G. Amorim), sc.ulhoa@gmail.com (S. C. Ulhoa), edilbertoo@gmail.com (Edilberto O. Silva)

called the symplectic form (sum over repeated indices is assumed). Consider the following bidifferential operator on  $C^\infty(T^*M)$ :

$$\Lambda = \frac{\overleftarrow{\partial}}{\partial q^\mu} \frac{\overrightarrow{\partial}}{\partial p_\mu} - \frac{\overleftarrow{\partial}}{\partial p^\mu} \frac{\overrightarrow{\partial}}{\partial q_\mu}, \quad (3)$$

such that for  $C^\infty$  functions,  $f = f(q^\mu, p^\mu)$  and  $g = g(q^\mu, p^\mu)$ , we have

$$\{f, g\} = \omega(f\Lambda, g\Lambda) = f\Lambda g, \quad (4)$$

where

$$\{f, g\} = \frac{\partial f}{\partial q^\mu} \frac{\partial g}{\partial p_\mu} - \frac{\partial f}{\partial p^\mu} \frac{\partial g}{\partial q_\mu}. \quad (5)$$

is the Poisson bracket and  $f\Lambda$  and  $g\Lambda$  are two vector fields given by  $h\Lambda = X_h = -\{h, \cdot\}$ . The space  $T^*$  endowed with this symplectic structure is called the phase space, and will be denoted by  $\Gamma$ .

The notion of Hilbert space associated with the phase space  $\Gamma$  is introduced by considering the set of square integrable functions,  $\phi(q^\mu, p^\mu)$  in  $\Gamma$ , such that

$$\int dp^\mu dq^\mu \phi^*(q^\mu, p^\mu) \phi(q^\mu, p^\mu) < \infty. \quad (6)$$

Then, we can write  $\phi(q^\mu, p^\mu) = \langle q^\mu, p^\mu | \phi \rangle$ , with

$$\int dp^\mu dq^\mu |q^\mu, p^\mu\rangle \langle q^\mu, p^\mu| = 1, \quad (7)$$

to be  $\langle \phi |$  the dual vector of  $|\phi\rangle$ . We call this the Hilbert space  $\mathcal{H}(\Gamma)$ .

### 3. Poincaré Algebra and Dirac Equation in Phase Space

Using the *star*-operators,  $\hat{A} = a_W(q, p) \star$ , we define 4-momentum and 4-position operators, respectively, by

$$\hat{P}^\mu = p^\mu \star = p^\mu \exp\left[\frac{i}{2}\left(\frac{\overleftarrow{\partial}}{\partial q^\mu} \frac{\overrightarrow{\partial}}{\partial p_\mu} - \frac{\overleftarrow{\partial}}{\partial p^\mu} \frac{\overrightarrow{\partial}}{\partial q_\mu}\right)\right] = p^\mu - \frac{i}{2} \frac{\partial}{\partial q_\mu}, \quad (8)$$

$$\hat{Q}^\mu = q^\mu \star = q^\mu \exp\left[\frac{i}{2}\left(\frac{\overleftarrow{\partial}}{\partial q^\mu} \frac{\overrightarrow{\partial}}{\partial p_\mu} - \frac{\overleftarrow{\partial}}{\partial p^\mu} \frac{\overrightarrow{\partial}}{\partial q_\mu}\right)\right] = q^\mu + \frac{i}{2} \frac{\partial}{\partial p_\mu}. \quad (9)$$

From Eqs. (8) and (9), we can introduce the quantity  $\hat{M}_{\nu\sigma} = \hat{Q}_\mu \hat{P}_\sigma - \hat{Q}_\sigma \hat{P}_\mu$ . The operators  $\hat{P}^\mu$  and  $\hat{M}_{\nu\sigma}$  are defined in the Hilbert space,  $\mathcal{H}(\Gamma)$ , constructed with complex functions in the phase space  $\Gamma$ , and satisfy the set of commutation relations

$$[\hat{M}_{\mu\nu}, \hat{P}_\sigma] = i(g_{\nu\sigma} \hat{P}_\mu - g_{\sigma\mu} \hat{P}_\nu), \quad (10)$$

$$[\hat{P}_\mu, \hat{P}_\nu] = 0, \quad (11)$$

$$[\hat{M}_{\mu\nu}, \hat{M}_{\sigma\rho}] = -i(g_{\mu\rho} \hat{M}_{\nu\sigma} - g_{\nu\rho} \hat{M}_{\mu\sigma} + g_{\mu\sigma} \hat{M}_{\rho\nu} - g_{\nu\sigma} \hat{M}_{\rho\mu}). \quad (12)$$

This is the Poincaré algebra, where  $\hat{M}_{\mu\nu}$  stands for rotations and  $\hat{P}_\mu$  for translations (but notice, in phase space). The Casimir invariants are calculated by using the Pauli-Lubanski matrices,  $\hat{W}_\mu = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \hat{M}^{\nu\sigma} \hat{P}^\rho$ , where  $\epsilon_{\mu\nu\rho\sigma}$  is the Levi-Civita symbol. The invariants are

$$\hat{P}^2 = \hat{P}^\mu \hat{P}_\mu, \quad (13)$$

and

$$\hat{W}^2 = \hat{W}^\mu \hat{W}_\mu, \quad (14)$$

where  $\hat{P}^2$  stands for the mass shell condition and  $\hat{W}^2$  for the spin.

To determine the Klein-Gordon field equation, we consider a scalar representation in  $\mathcal{H}(\Gamma)$ . In this case, we can use the invariant  $\hat{P}^2$  to write

$$\begin{aligned} \hat{P}^2 \phi(q^\mu, p^\mu) &= (p^2) \star \phi(q^\mu, p^\mu), \\ &= (p^\mu \star p_\mu) \phi(q^\mu, p^\mu), \\ &= m^2 \phi(q^\mu, p^\mu), \end{aligned} \quad (15)$$

where  $m$  is a constant fixing the representation and interpreted as mass, such that the mass shell condition is satisfied. Using Eq. (8), we obtain

$$\left(p^\mu p_\mu - i p^\mu \frac{\partial}{\partial q^\mu} - \frac{1}{4} \frac{\partial}{\partial q^\mu} \frac{\partial}{\partial q_\mu}\right) \phi(q^\mu, p^\mu) = m^2 \phi(q^\mu, p^\mu), \quad (16)$$

which is the Klein-Gordon equation in phase space.

The association of this representation with Wigner formalism is given by [22]

$$f_W(q^\mu, p^\mu) = \phi(q^\mu, p^\mu) \star \psi^*(q^\mu, p^\mu), \quad (17)$$

where  $f_W(q^\mu, p^\mu)$  is the relativistic Wigner function.

The representation for spin-1/2 leads to

$$\gamma^\mu \left(p_\mu - \frac{i}{2} \frac{\partial}{\partial q^\mu}\right) \psi(q^\mu, p^\mu) = m \psi(q^\mu, p^\mu), \quad (18)$$

which is the Dirac equation in phase space, where the  $\gamma^\mu$ -matrices fulfill the usual Clifford algebra,  $(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) = 2g^{\mu\nu}$ . The Wigner function, in this case, is given by [22]

$$f_W(q^\mu, p^\mu) = \psi(q^\mu, p^\mu) \star \bar{\psi}(q^\mu, p^\mu), \quad (19)$$

where  $\bar{\psi}(q^\mu, p^\mu) = \gamma^0 \psi^\dagger(q^\mu, p^\mu)$ , with  $\psi^\dagger(q^\mu, p^\mu)$  being the Hermitian of  $\psi(q^\mu, p^\mu)$ . We point out that the CPT theorem holds for non-commutative theories as showed in [25]. Therefore, such a theorem is also valid in phase space since the group structure remains the same.

One central point to be emphasized is that the approach developed here permits the calculation of Wigner functions for relativistic systems with methods, based on symmetry, similar to those used in quantum field theory.

### 4. Solution of Dirac Equation with Electromagnetic Interaction on Phase Space

In this section, we study interactions of a spin-1/2 charged particle with an external electromagnetic field in Phase Space. The relevant equation is the Dirac equation with minimal coupling

$$(\gamma^\mu \hat{P}_\mu + m) \Psi = 0, \quad (20)$$

being

$$\hat{P}_\mu \rightarrow \hat{P}_\mu - e \hat{A}_\mu, \quad (21)$$

the minimal coupling prescription, where  $\hat{A}^i = \frac{1}{2}\epsilon^{ijk}B_j\hat{X}_k$  and  $\hat{A}^0 = 0$ , which represents the chosen gauge. We also chose the magnetic field as  $\mathbf{B} = (0, 0, B)$ . Thus, we have

$$[\gamma^\mu (\hat{P}_\mu - e\hat{A}_\mu) + m]\Psi = 0. \quad (22)$$

Now, we make the definition

$$\Psi = [\gamma^\mu (\hat{P}_\mu - e\hat{A}_\mu) - m]\psi = 0. \quad (23)$$

In order to obtain the energy levels, we substitute Eq. (23) into Eq. (22) to give

$$[\gamma^\mu \gamma^\nu (\hat{P}_\mu - e\hat{A}_\mu)(\hat{P}_\nu - e\hat{A}_\nu) - m^2]\psi = 0, \quad (24)$$

with

$$\gamma^\mu \gamma^\nu = g^{\mu\nu} + \sigma^{\mu\nu}, \quad (25)$$

where

$$\sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) = \frac{i}{2}[\gamma^\mu, \gamma^\nu]. \quad (26)$$

The components  $\sigma^{0i}$ ,  $\sigma^{ij}$  of the operator (26) are

$$\sigma^{0i} = i \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad \sigma^{ij} = - \begin{pmatrix} \epsilon_{ijk}\sigma^k & 0 \\ 0 & \epsilon_{ijk}\sigma^k \end{pmatrix}. \quad (27)$$

Note that these components are also expressed as  $\sigma^{0j} = i\alpha^j$ ,  $\sigma^{ij} = -\epsilon_{ijk}\Sigma^k$ . These results are explicitly evaluated in the following representation of the  $\gamma$ -matrices:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad \Sigma^k = \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}. \quad (28)$$

with  $\sigma = (\sigma_x, \sigma_y, \sigma_z)$  being the Pauli matrices. Equation (24) can be written as

$$\{\hat{P}^\mu \hat{P}_\mu - e(\hat{P}^\mu \hat{A}_\mu + \hat{A}^\mu \hat{P}_\mu) + e^2 \hat{A}^\mu \hat{A}_\mu + e\sigma^{\mu\nu}[\hat{P}^\nu, \hat{A}_\mu] - m^2\}\psi = 0. \quad (29)$$

Confining the motion at the plane  $\hat{X}\hat{Y}$  by the choice  $\hat{P}_3 = 0$  and using the operators  $\hat{P}_\mu = p_\mu - \frac{i}{2}\frac{\partial}{\partial x^\mu}$  and  $\hat{X}_\mu = x_\mu - \frac{i}{2}\frac{\partial}{\partial p^\mu}$ , we get the following equation

$$\begin{aligned} & \left(-E^2 - iE\frac{\partial}{\partial t} + \frac{1}{4}\frac{\partial^2}{\partial t^2} - m^2\right)\psi + \left\{p_x^2 + p_y^2 - \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \right. \\ & - eB\left[\frac{i}{2}\left(p_y\frac{\partial}{\partial p_x} - p_x\frac{\partial}{\partial p_y}\right) + \frac{1}{4}\left(\frac{\partial^2}{\partial y\partial p_x} - \frac{\partial^2}{\partial x\partial p_y}\right)\right] \\ & - i\left(p_y\frac{\partial}{\partial y} - p_x\frac{\partial}{\partial x}\right) - eB\left[(xp_y - yp_x) - \frac{i}{2}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right] \\ & \left. + \frac{e^2 B^2}{4}\left[\left(x + \frac{i}{2}\frac{\partial}{\partial p_x}\right)^2 + \left(y + \frac{i}{2}\frac{\partial}{\partial p_y}\right)^2\right] + ieB\sigma^{12}\right\}\psi = 0. \quad (30) \end{aligned}$$

Since  $\gamma_5$  commutes through all terms of Eq. (30) and, if we have found the solution  $\psi$ , then we must also have that  $\gamma_5\psi$  is a solution. In this case, the wave function can take on one of the forms

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ -\psi_1 \\ -\psi_2 \end{pmatrix}. \quad (31)$$

Thus, we can select only one of these solutions as that will make the other redundant. We specialize the solution

$$\gamma_5\psi = \psi. \quad (32)$$

We can write  $\psi$  in terms of  $\Psi$  in Eq. (23) as

$$\psi = \frac{1}{2}(I + \gamma_5)\Psi, \quad (33)$$

where  $I$  is the unit matrix. From Eq. (31), we can show that Eq. (33) can be put in the form

$$\psi = \begin{pmatrix} \chi(E, t, p_x, p_y, x, y) \\ -\chi(E, t, p_x, p_y, x, y) \end{pmatrix} = \begin{pmatrix} \varphi(E, t)\phi(p_x, p_y, x, y) \\ -\varphi(E, t)\phi(p_x, p_y, x, y) \end{pmatrix}, \quad (34)$$

where  $\chi(E, t, p_x, p_y, x, y)$  is a two-component wavefunction. Note that, in the representation (34), the upper two components of Eq. (30) are now completely decoupled from the lower two. So, we have, in two-component form, the following equations:

$$\left(-E^2 - iE\frac{\partial}{\partial t} + \frac{1}{4}\frac{\partial^2}{\partial t^2} - m^2\right)\varphi = \lambda^2\varphi, \quad (35)$$

$$\begin{aligned} & \left\{p_x^2 + p_y^2 - \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - i\left(p_y\frac{\partial}{\partial y} - p_x\frac{\partial}{\partial x}\right) \right. \\ & - eB\left[(xp_y - yp_x) + \frac{i}{2}\left(p_y\frac{\partial}{\partial p_x} - p_x\frac{\partial}{\partial p_y}\right) \right. \\ & - \frac{i}{2}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) + \frac{1}{4}\left(\frac{\partial^2}{\partial y\partial p_x} - \frac{\partial^2}{\partial x\partial p_y}\right)] \\ & \left. + \frac{e^2 B^2}{4}\left[\left(x + \frac{i}{2}\frac{\partial}{\partial p_x}\right)^2 + \left(y + \frac{i}{2}\frac{\partial}{\partial p_y}\right)^2\right] + ieB\sigma^{12}\right\}\phi = \lambda^2\phi, \quad (36) \end{aligned}$$

where  $\lambda$  is a constant. We point out that  $E$  is not associated to  $i\frac{\partial}{\partial t}$  a priori in Eq. 35, thus the energy comes from  $\lambda$ . Note that Eqs. (35) and (36) determines  $\chi$  and, hence, from Eq. (33), it determines  $\psi$ . Performing a changing of variables in Eq. (36) of the form

$$z = p_x^2 + p_y^2 + eB(y p_x - x p_y) + \frac{e^2 B^2}{4}(x^2 + y^2),$$

we note that the imaginary part of the equation vanished which yields

$$z\phi - e^2 B^2 \dot{\phi} - e^2 B^2 z \ddot{\phi} = (\lambda^2 + seB)\phi, \quad (37)$$

where we have used  $i\sigma^{12}\phi = -s\phi$ , with  $s = \pm 1$ . If we use  $\omega = z/eB$  and  $\phi = \exp(-\omega)F(\omega)$ , the equation for  $F(\omega)$  is found to be

$$\omega F'' + (1 - 2\omega)F' - (1 - k)F = 0, \quad (38)$$

where  $F' \equiv \frac{dF}{d\omega}$  and  $k = (\lambda^2 + seB)/eB$ . Equation (38) is of the confluent hypergeometric equation type

$$zF''(z) + (b - z)F'(z) - aF(z) = 0. \quad (39)$$

In this manner, the general solution for Eq. (38) is given by

$$F(\omega) = A_m M\left(\frac{1}{2} - \frac{k}{2}, 1, 2\omega\right) + B_m U\left(\frac{1}{2} - \frac{k}{2}, 1, 2\omega\right), \quad (40)$$

where  $M(a, b, z)$ ,  $U(a, b, z)$  are the Kummer functions, and  $A_m$ ,  $B_m$  are constants. Since only  $U(a, b, z)$  is square integrable, we consider it as a physical solution. Thus, we can impose that  $A_m = 0$ . Furthermore, if  $a = -n$ , with  $n = 0, 1, 2, \dots$ , the series  $U(a, b, z)$  becomes a polynomial in  $z$  of degree not exceeding  $n$ . From this condition, we can write

$$1 - k = -2n, \quad (41)$$

from which, we can extract the relation

$$\lambda^2 = eB(2n + 1 + s). \quad (42)$$

The wave function is given by

$$f_m(z) = B_m M\left(-n, 1, \frac{2z}{eB}\right), \quad (43)$$

where  $B_m$  is a normalization constant.

The Wigner function related to Dirac equation with an electromagnetic interaction is formally given by

$$f_W(x, y, p_x, p_y) = \Psi_n(x, y, p_x, p_y) \star \bar{\Psi}_n(x, y, p_x, p_y).$$

Thus Wigner function is used to determine mean values, for example, this result can be useful for theoretical and applied areas, such as: quantum optics, quantum tomography and quantum computing. We point out that Landau levels which appear in expression (42) represent as a matter of fact a planar oscillator and the variable  $z$  in Eq. (43) give us information about the symplectic structure.

## 5. Conclusion

We have set forth a symplectic representation of the Poincaré group, which yields quantum theories in phase space. We have derived the Klein-Gordon and Dirac equations in phase space and, as illustrations, studied the Dirac equation with electromagnetic interaction. The symplectic representation is constructed on the basis of the Moyal or star product, an ingredient of noncommutative geometry. A Hilbert space is then defined from a manifold with the features of phase space. The states are represented by a quasi-amplitude of probability, a wave function in phase space, the definition of which makes connection with the Wigner function, i.e., the quasi-probability density. Nontrivial, yet consistent, the association with the Wigner function provides a physical interpretation of the theory. Analogous interpretations are not found in other studies of representations in phase space [25, 26]. One aspect of the procedure deserves emphasis. Our formalism explores unitary representations to calculate Wigner functions. This constitutes an important advantage over the more traditional constructions of the Wigner method, which entail several intricacies associated with the Liouville-von Neumann equation. Furthermore, the formalism we have described opens new perspectives for applications of the Wigner function method in quantum field theory. This aspect of the formalism will be discussed in a forthcoming paper.

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